Monodromy representations and the Knizhnik-Zamolodchikov equation

ref: ch5 of Ohtsuki

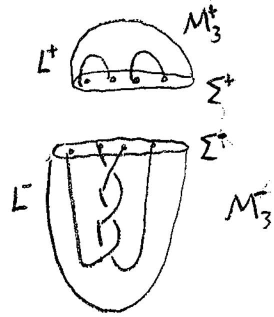
physics background / motivation (ref, roughly: Witten '89 QFT and Jones polynomial; appendix F (especially F.4) of Ohtsuki)

3d TQFT - Chern Simons theory w/ gauge group G

"expectation value" of "Wilson loop" inserted along L is quantum invariant

$$Z_G(W_{L;R}; M_3)$$

$\Big\downarrow$ slice M_3 along surface Σ transverse to L



$L^+ \subset M_3^+$, $L^- \subset M_3^-$ each correspond to "quantum state" and inner product recovers quantum invariant:

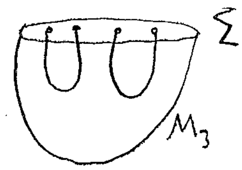
$$Z(W_{L;R}; M_3) = \langle \psi_{L^+, M_3^+}, \psi_{L^-, M_3^-} \rangle$$

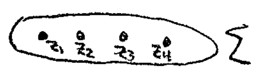
correspondence between:

states in 3d CS on M_3 w/ boundary Σ $\longleftrightarrow$ states in 2d WZW CFT on Σ

$\downarrow$ "correlation functions"

$$\langle \phi_{R_1}(z_1)\, \phi_{R_2}(z_2)\, \phi_{R_3}(z_3)\, \phi_{R_4}(z_4) \rangle$$





Wilson line $\longleftrightarrow$ operator inserted at pt

n-point correlation functions: (1) depend on n pts $z_i \in \Sigma$

(2) form vector space

(3) satisfy differential eqn called KZ eqn

KZ eqn

Lie algebra background

$\mathfrak{g}$ semi-simple Lie algebra / $\mathbb{C}$

adjoint rep $ad: \mathfrak{g} \to End(\mathfrak{g})$ defined by $(ad(X))(Y) = [X,Y]$

Killing form $B: \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ defined by $tr(ad(X)\,ad(Y))$

choose orthonormal basis $\{I_\mu\}$ of $\mathfrak{g}$ w.r.t. B ie $B(I_\mu, I_\nu) = \delta_{\mu\nu}$

using dual basis $\{I_\mu^*\}$ of $\mathfrak{g}^*$, we can view B as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^*$

define $\Upsilon \in \mathfrak{g} \otimes \mathfrak{g}$ as the dual of B

$\Rightarrow$ $\Upsilon = \sum_\mu I_\mu \otimes I_\mu$ (up to rescaling)

ex: $\mathfrak{g} = sl_2$

$$\Upsilon = E \otimes F + F \otimes E + \tfrac{1}{2} H \otimes H$$

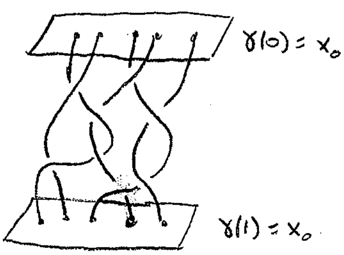
define
$$\Upsilon_{ij} := \sum_\mu 1 \otimes \ldots \otimes 1 \otimes \underset{\substack{\uparrow \\ i\text{th position}}}{I_\mu} \otimes 1 \otimes \ldots \otimes 1 \otimes \underset{\substack{\uparrow \\ j\text{th position}}}{I_\mu} \otimes 1 \otimes \ldots \otimes 1 \in U(\mathfrak{g})^{\otimes n}$$

KZ eqn

Let X_n be configuration space of n distincts pts in $\mathbb{C}$

$$X_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for any } i \neq j\}$$

Fix a base pt $x_0 \in X_n$ and consider paths $\gamma: [0,1] \to X_n$ s.t. $\gamma(0) = \gamma(1) = x_0$



two paths are homotopic iff braids are homotopic

$\Rightarrow$ $\pi_1(X_n) = P_n$ (pure braid group)

$\Rightarrow$ $\pi_1(X_n / S_n) = B_n$ (braid group)

Let V be a v.s. with action of $U(\mathfrak{g})$

so $T_{ij} \in U(\mathfrak{g})^{\otimes n}$ acts on $V^{\otimes n}$

$W: X_n \to V^{\otimes n}$ smooth function

KZ eqn is

$$\frac{\partial W}{\partial z_i} = \frac{\hbar}{2\pi i} \sum_{1 \le i < j \le n} \frac{T_{ij}}{z_i - z_j} W$$

or equiv.

$$dW = \underbrace{\frac{\hbar}{2\pi i} \sum_{1 \le i < j \le n} T_{ij} \frac{dz_i - dz_j}{z_i - z_j}}_{\tau_n} W = \tau_n W$$

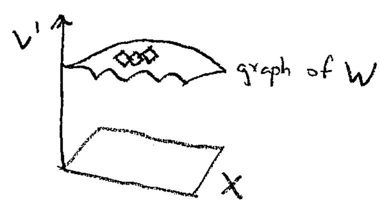
Prop A non-commutative algebra
$\curvearrowright$
V' v.s.
α A-valued 1-form on X
$W: X \to V'$

If $d\alpha - \alpha \wedge \alpha = 0$ then $(d-\alpha)W = 0$ has unique local solution W in nbhd of any pt $x_0 \in X$ w/ arbitrary given initial value $v = W(x_0) \in V'$

such an α gives a flat connection $d - \alpha$ on trivial vector bundle over X w/ fiber V'

Prop τ_n satisfies $d\tau_n - \tau_n \wedge \tau_n = 0$

Prop $\Rightarrow$ integrating along $d - \alpha$ gives local section of vector bundle

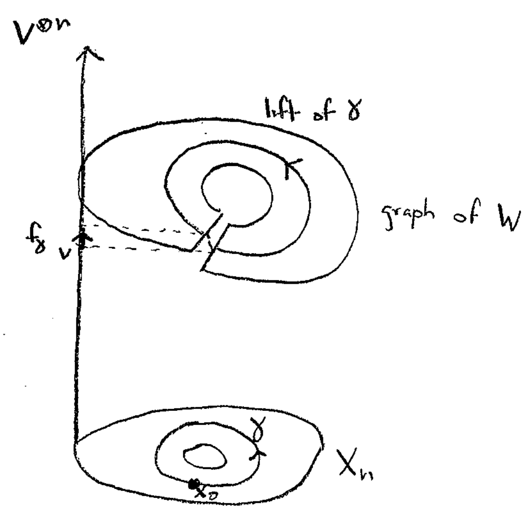


closed path $\gamma: [0,1] \to X_n$ s.t. $\gamma(0) = \gamma(1) = x_0$

By Prop, obtain a path $f_\gamma: [0,1] \to V^{\otimes n}$
locally given by $f_\gamma(t) = W(\gamma(t))$ with $f_\gamma(0) = v$

$f_\gamma(t)$ is invariant under homotopy of γ

$\Rightarrow$ closed path γ gives a map $\rho_{KZ}(\gamma): V^{\otimes n} \to V^{\otimes n}$
taking $v = f_\gamma(0)$ to $f_\gamma(1)$



$\rho_{KZ}: P_n \to End(V^{\otimes n})$ is rep of P_n

If we instead take path $\tilde{\gamma} \in X_n/S_n$, we get rep of B_n
$\rho_{KZ}: B_n \to End(V^{\otimes n})$

called <u>monodromy rep</u> of B_n

computing monodromies of KZ eqn

n=2 case

$$dW = \frac{\hbar \tau}{2\pi i} \frac{dz_1 - dz_2}{z_1 - z_2} W$$

local soln $W(z_1, z_2) = (z_2 - z_1)^{\hbar\tau/2\pi i} v$ for $v \in V \otimes V$ defined for, say, $z_2 - z_1 \in \mathbb{C} - (-\infty, 0]$

define $\gamma \in X_2$ by $\gamma(t) = (0, e^{2\pi i t})$ for $t \in [0,1]$

then $W(0, e^{2\pi i t}) = e^{\hbar \tau t} v$

so along γ_t, v is taken to $e^{\hbar\tau}$ ie $\rho_{KZ}: P_2 \to End(V^{\otimes 2})$ is given by

$\rho_{KZ}(\sigma^2) = e^{\hbar \tau}$

using $\gamma \in X_2/S_2$ given by $\gamma(t) = (0, e^{\pi i t})$ for $t \in [0,1]$, we find $\rho_{KZ}: B_2 \to End(V^{\otimes 2})$ given by

$\rho_{KZ}(\sigma) = P \circ e^{\hbar\tau/2}$

where P is permutation on $V^{\otimes 2}$

n=3 case

$$dW = \frac{\hbar}{2\pi i}\left(\tau_{12}\, d\log(z_1 - z_2) + \tau_{13}\, d\log(z_1 - z_3) + \tau_{23}\, d\log(z_2 - z_3)\right) W$$

one can show there are two solns

$$W_{(\circ\circ)\circ}(z_1, z_2, z_3) = f\left(\frac{z_2 - z_1}{z_3 - z_1}\right)(z_2 - z_1)^{\hbar\tau_{12}/2\pi i}(z_3 - z_1)^{\hbar(\tau_{13} + \tau_{23})/2\pi i}$$

$$W_{\circ(\circ\circ)}(z_1, z_2, z_3) = g\left(\frac{z_3 - z_2}{z_3 - z_1}\right)(z_3 - z_2)^{\hbar\tau_{23}/2\pi i}(z_3 - z_1)^{\hbar(\tau_{12} + \tau_{13})/2\pi i}$$

where $f(z), g(z) \in \mathbb{C}\langle\langle \hbar\tau_{12}, \hbar\tau_{23}\rangle\rangle$ are analytic fcns w/ $f(0) = g(0) = 1$ defined in nbhd of $0 \in \mathbb{C}$

solns are related by $W_{(\circ\circ)\circ} = W_{\circ(\circ\circ)}\, \varphi_{KZ}(\hbar\tau_{12}, \hbar\tau_{23})$

$\underbrace{\varphi_{KZ}(\hbar\tau_{12}, \hbar\tau_{23})}_{\Phi_{KZ}}$ is Drinfel'd associator consider rep $\rho_{KZ}: P_3 \longrightarrow \mathrm{End}(V^{\otimes 3})$ for basept $(0, \varepsilon, 1) \in X_3$

path $\gamma(t) = (0, \varepsilon e^{2\pi i t}, 1) \in X_3$ gives $W_{(\circ\circ)\circ}(0, \varepsilon e^{2\pi i t}, 1) = f(\varepsilon e^{2\pi i t})\, \varepsilon^{\hbar\Upsilon_{12}/2\pi i}\, e^{\hbar\Upsilon_{12} t}$

$\Rightarrow$ monodromy along γ takes $f(\varepsilon)\, \varepsilon^{\hbar\Upsilon_{12}/2\pi i} v \in V^{\otimes 3}$ to $f(\varepsilon)\, \varepsilon^{\hbar\Upsilon_{12}/2\pi i} e^{\hbar\Upsilon_{12}} v$

$\Rightarrow$ $\rho_{KZ}(\sigma_1^2) = f(\varepsilon)\, \varepsilon^{\hbar\Upsilon_{12}/2\pi i}\, e^{\hbar\Upsilon_{12}}\, \varepsilon^{-\hbar\Upsilon_{12}/2\pi i}\, f(\varepsilon)^{-1}$

next, consider path $\gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$ where

γ_1 is path from $(0, \varepsilon, 1)$ to $(0, 1-\varepsilon, 1)$ along $\mathbb{R}$

$\gamma_2 = (0, 1-\varepsilon e^{2\pi i t}, 1)$

monodromy along γ_1 takes $f(\varepsilon)\, \varepsilon^{\hbar\Upsilon_{12}/2\pi i} v$ to $f(1-\varepsilon)(1-\varepsilon)^{\hbar\Upsilon_{12}/2\pi i} v = g(\varepsilon)\, \varepsilon^{\hbar\Upsilon_{23}/2\pi i}\, \Phi_{KZ}$

monodromy along γ_2 is similar to monodromy along γ

altogether, monodromy along $\gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$ gives

$$\rho_{KZ}(\sigma_2^2) = f(\varepsilon)\, \varepsilon^{\hbar\Upsilon_{12}/2\pi i}\, \Phi_{KZ}\, e^{\hbar\Upsilon_{23}}\, \Phi_{KZ}\, \varepsilon^{-\hbar\Upsilon_{12}/2\pi i}\, f(\varepsilon)^{-1}$$

conjugate to obtain $\rho_{KZ}: P_3 \longrightarrow \mathrm{End}(V^{\otimes 3})$ as

$$\rho_{KZ}(\sigma_1^2) = e^{\hbar\Upsilon_{12}}$$
$$\rho_{KZ}(\sigma_2^2) = \Phi_{KZ}^{-1}\, e^{\hbar\Upsilon_{23}}\, \Phi_{KZ}$$

similarly, find $\rho_{KZ}: B_3 \longrightarrow \mathrm{End}(V^{\otimes 3})$ as

$$\rho_{KZ}(\sigma_1) = P_{12} \circ e^{\hbar\Upsilon_{12}/2}$$
$$\rho_{KZ}(\sigma_2) = \Phi_{KZ}^{-1}\left(P_{23} \circ e^{\hbar\Upsilon_{12}/2}\right) \Phi_{KZ}$$

<u>n=4 case</u>

5 solutions

$$W_{((\circ\circ)\circ)\circ} = f_1\left(\frac{z_2 - z_1}{z_3 - z_1}, \frac{z_3 - z_1}{z_4 - z_1}\right) \cdots$$

$$W_{(\circ(\circ\circ))\circ} = f_2\left(\frac{z_3 - z_2}{z_3 - z_1}, \frac{z_3 - z_1}{z_4 - z_1}\right) \cdots$$

$W_{\circ((\circ\circ)\circ)} = \cdots$

$W_{\circ(\circ(\circ\circ))} = \cdots$

$W_{(\circ\circ)(\circ\circ)} = \cdots$ related by φ_{KZ} ie $W_{((\circ\circ)\circ)\circ} = W_{(\circ(\circ\circ))\circ}\, \varphi_{KZ}(\hbar\Upsilon_{12}, \hbar\Upsilon_{23})$

analogs of previous arguments give $\rho_{KZ}: B_4 \to End(V^{\otimes 4})$ as

$$\rho_{KZ}(\sigma_1) = P_{12} \circ e^{\hbar T_{12}/2}$$

$$\rho_{KZ}(\sigma_2) = (\Phi_{KZ}^{-1} \otimes 1)(P_{23} \circ e^{\hbar T_{12}/2})(\Phi_{KZ} \otimes 1)$$

$$\rho_{KZ}(\sigma_3) = (\Delta \otimes 1 \otimes 1)\, \Phi_{KZ}^{-1} \cdot (P_{23} \circ e^{\hbar T_{12}/2}) \cdot (\Delta \otimes 1 \otimes 1)\, \Phi_{KZ}$$

where $\Delta: U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is $\Delta(X) = X \otimes 1 + 1 \otimes X$ for any $X \in \mathfrak{g}$

<u>general n case</u>

want to consider solns of KZ as solns near boundary of certain compactification of real configuration space

$$Y_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i \neq t_j \text{ for any } i \neq j\}$$

$$Y_n / S_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_1 < t_2 < \dots < t_n\}$$

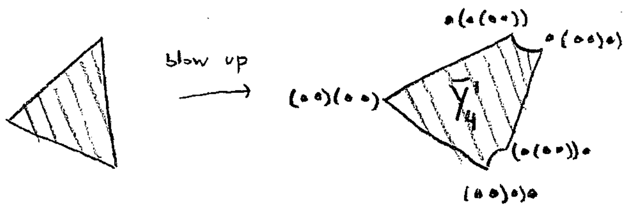
quotient by $t \mapsto at + b$ to get

$$Y_n' = \{(t_2, \dots, t_{n-1}) \in \mathbb{R}^n \mid 0 < t_2 < \dots < t_{n-1} < 1\}$$

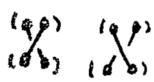
compactify by iterative real blow ups of $\{(t_2, \dots, t_{n-1}) \mid 0 \leq t_2 \leq \dots \leq t_{n-1} \leq 1\}$

<u>ex</u>: $n=4$

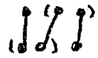
$$\bar{Y}_4' = \{(t_2, t_3; s_1, s_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid 0 < t_2 < t_3 < 1,\ s_1 = \frac{t_2}{t_3},\ s_2 = \frac{t_3 - t_2}{1 - t_2}\}$$

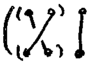


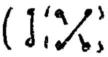
pts in bdry of $\bar{Y}_n'$ correspond to parenthesized sets of n dots and paths in $\bar{Y}_n'$ correspond to parenthesized bundles of vertical lines

also introduce parenthesized braids 

using this notation, we can re-express $\rho_{KZ}: B_3 \to End(V^{\otimes 3})$ as

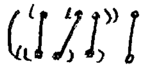
$\rho_{KZ}($  $) = \Phi_{KZ}$

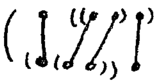
$\rho_{KZ}($  $) = (P \circ R_{KZ}) \otimes 1$

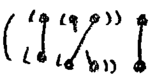
$\rho_{KZ}($  $) = 1 \otimes (P \circ R_{KZ})$

for $R_{KZ} := e^{t\gamma/2}$

similarly, $\rho_{KZ}: B_4 \to End(V^{\otimes 4})$ can be built using

$\rho_{KZ}($  $) = \Phi_{KZ} \otimes 1$

$\rho_{KZ}($  $) = (id \otimes \Delta \otimes id)\, \Phi_{KZ}$

$\rho_{KZ}($  $) = 1 \otimes (P \circ R_{KZ}) \otimes 1$

etc....

by extending these formulae with $\otimes 1$ and Δ, we can construct $\rho_{KZ}: B_n \to End(V^{\otimes n})$