Integrality of WRT invariants

"Integrality of quantum 3-man. inv. and rational surgery formula"
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- Thm The $SO(3)$ inv. $\tau_M \in G_N = \dfrac{\mathbb{Z}[q]}{\Phi_N(q)}$ for N odd.

We'll assume $N=p$ prime.

I) Definition

$\{n\} = q^{n/2}-q^{-n/2}$ $\quad \{n\}! = \prod_{i=1}^{n} \{i\}$ $\quad [n] = \dfrac{\{n\}}{\{1\}}$ $\quad \begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{\{n\}!}{\{k\}!\{n-k\}!}$

$(q)_n = (q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$. $\quad (q^2)_n = (1-q^2)(1-q^4)\cdots(1-q^{2n})$.

Let V_n be U_q-rep of dim n

Let $J_L(V_n)$ be colored Jones poly of L where L is oriented, framed link in S^3

$J_U(V_n) = [n]$ $\qquad q^{1/4} J_{L^+} - q^{-1/4} J_{L^-} = (q^{1/2}-q^{-1/2}) J_{L^0}$

If L' is obtained from L by inc. framing by 1, $J_{L'}(n) = q^{\frac{n^2-1}{4}} J_L(n)$.

Let $F_L = \displaystyle\sum_{\substack{n_i=1 \\ \text{odd}}}^{2p-1} \prod_{i=1}^{m} [n_i] J_L(n_1,\dots,n_m)$ where L's comp. colored by $n_1,\dots,n_m$

Let M be closed oriented 3-man. obtained by surgery on L.

Then $\tau_M = \dfrac{F_L}{(F_{U^+})^{\sigma^+}(F_{U^-})^{\sigma^-}}$ where U^+:  U^-: 

$\sigma^\pm$ are the # of $\pm$ eigenvalues of linking matrix

Note $\tau_{M\#N} = \tau_M \tau_N$.

II) Alg. facts

- Lemma[1]: If $k > \frac{p-3}{2}$, then $\frac{\{2k+1\}!}{\{k\}!\{1\}} = 0$

Pf: Uses fact that $\{p\} = 0$

- Gauss sums:

$$\gamma_d := \sum_{\substack{n=1 \\ odd}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} = \begin{cases} q^{-\frac{d}{4}} \left(\frac{d}{p}\right)\sqrt{p} & \text{if } p \equiv 1 \mod 4 \\ q^{-\frac{d}{4}} \left(\frac{d}{p}\right) i\sqrt{p} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Lemma[2]: $\displaystyle\sum_{\substack{n=1 \\ odd}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} q^{na} = \gamma_d \cdot q^{-a^2/d}$ where $a \in \mathbb{Z}$.

Pf: Complete square

- Lemma[3]: $F_{U^{\pm 1}} = \mp 2\, \gamma_{\pm 1} \dfrac{q^{\mp 1/2}}{\{1\}}.$

Pf: Use previous lemma

So $F_{U^{\operatorname{sgn} d}} = -2\, \gamma_{\operatorname{sgn} d} \dfrac{q^{-\operatorname{sgn} d \cdot 1/2}}{\{1\}}$

- Lemma[4]: $\dfrac{\gamma_d}{\gamma_{\operatorname{sgn} d}} = \left(\dfrac{|d|}{p}\right) q^{\frac{\operatorname{sgn} d - d}{4}}$ where $\left(\dfrac{|d|}{p}\right) = \begin{cases} 0 & \text{if } |d| \equiv 0 \mod p \\ 1 & \text{if } |d| \equiv x^2 \mod p \\ -1 & \text{if } |d| \not\equiv x^2 \mod p. \end{cases}$

Legendre symbol

Pf: Use previous lemmas

• Let $H(k,b) = q^{\frac{(k+1)(k+2)}{4}} \sum_{k+1 \geq n_b \geq n_{b-1} \geq \cdots \geq n_1 \geq 0} q^{n_1^2+\cdots+n_b^2} \frac{(q)_{k+1}}{\prod_{i=1}^{b} (q)_{n_{i+1}-n_i}}$

Let $\mathcal{G}(k,b) = \sum_{j=0}^{2k+1} (-1)^j \begin{bmatrix} 2k+1 \\ j \end{bmatrix} q^{b(j-k)^2}$

• Lemma[5]: Suppose $bd \equiv 1 \bmod p$. Then

$$\sum_{\substack{n=1 \\ \text{odd}}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{n\}! \, \{n\} = -2\gamma_d \frac{\mathcal{G}(k,-b)}{\frac{\{2k+1\}!}{\{k\}!}}$$

• Prop[6]: $\dfrac{\mathcal{G}(k,b)}{\frac{\{2k+1\}!}{\{k\}!}} = -\operatorname{sgn} b \; H(k,b)$

Pf: Tricky using identities of q-series

III) Integrality when L is a knot

- Suppose L has framing $d \in \mathbb{Z}$.

Let $L_0 = L$ with 0 framing. Then $J_L(n) = q^{d\left(\frac{n^2-1}{4}\right)} J_{L_0}(n)$

$$\text{so } \tau_M = \sum_{\substack{n=1\\ \text{odd}}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} [n]\, J_{L_0}(n) \Big/ F_U^{\,\text{sgn}\, d}$$

- Habiro: Let $P_k = \prod_{i=1}^{k} \left(V_r - \left(q^{\frac{2i+1}{2}} - q^{-\left(\frac{2i+1}{2}\right)}\right)\right)$, $P_k' = P_k / \{k\}!$

$$V_n = \sum_{k=0}^{n-1} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} P_k = \sum_{k=0}^{n-1} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\, P_k'$$

$$\text{so } J_L(n) = \sum_{k=0}^{n-1} J_L(P_k') \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!$$

Thm (Habiro): $J_L(P_k') \in \begin{bmatrix} 2k+1 \\ k \end{bmatrix} (q^2)_k\, \mathbb{Z}[q^{\pm 1/4}]$.

$$\cdot\ \tau_M = \sum_{\substack{n=1\\ \text{odd}}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} [n]\, J_{L_0}(n) \Big/ F_U^{\,\text{sgn}\, d}$$

$$= \sum_{\substack{n=1\\ \text{odd}}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} \sum_{k=0}^{p-3} J_{L_0}(P_k') \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\, [n] \Big/ F_U^{\,\text{sgn}\, d}$$

$$= \sum_{k=0}^{p-3} J_{L_0}(P_k') \sum_{\substack{n=1\\ \text{odd}}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\, [n] \Big/ F_U^{\,\text{sgn}\, d}$$

$$= \frac{\displaystyle\sum_{k=0}^{p-3} J_{L_0}(P_k') \sum_{\substack{n=1\\ \text{odd}}}^{2p-1} q^{d\left(\frac{n^2-1}{4}\right)} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\, [n]}{\dfrac{-2\, \gamma_{\text{sgn}\, d}\, q^{-\text{sgn}\, d \cdot 1/4}}{\{1\}}}$$

$$= \frac{-q^{\operatorname{sgn} d \cdot \frac{1}{2}} \sum_{k=0}^{\frac{p-3}{2}} J_{L_0}(P_k') \sum_{\substack{n=1 \\ \text{odd}}}^{p+1} q^{d\left(\frac{n^2-1}{24}\right)} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}! \{n\}}{2\gamma_{\operatorname{sgn} d}}$$

Lemma 5
Prop 1

$$= \frac{-q^{\operatorname{sgn} d \cdot \frac{1}{2}} \sum_{k=0}^{\frac{p-3}{2}} J_{L_0}(P_k') \left(-2 \operatorname{sgn} b \; \gamma_d \, H(k, -b)\right)}{2\gamma_{\operatorname{sgn} d}}$$

where $b \in \mathbb{Z}$ s.t. $bd = 1 \mod p$.

Lemma 4

$$= \operatorname{sgn} b \; q^{1/2 \operatorname{sgn} d} \sum_{k=0}^{\frac{p-3}{2}} J_{L_0}(P_k') \, H(k, -b) \cdot \left(\frac{|d|}{p}\right) q^{\frac{\operatorname{sgn} d - d}{4}}$$

$\in \mathcal{O}_p.$

IV) General case

- If L is alg. split (linking matrix is diagonal), result follow similarly.
- If L isn't alg. split, then $\exists$ connected sum of lens spaces $N = \#_{j=1}^{r} L(2^{k_j}, -1)$ s.t. $M \# M \# N$ is given by surgery on split link.

Then $\tau_M^2 \in \mathcal{O}_p$. Thus $\tau_M \in \mathcal{O}_p$ since $\mathcal{O}_p$ is an integral domain.

Appendix A: Gauss Sums

· Let $G(p) = \sum_{n=0}^{p-1} q^{n^2}$ $\qquad \hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \qquad \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(k)$

$= \sqrt{p}$ if $p \equiv 1 \mod 4$

$= i\sqrt{p}$ if $p \equiv 3 \mod 4$

Let $f(x) = \begin{cases} q^{x^2} & \text{if } 0 \le x < p \\ 0 & \text{otherwise} \end{cases}$

Then $G(p) = \sum_{n=0}^{p-1} f(n) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k) = \sum_{k\in\mathbb{Z}} \int_0^p e^{\frac{2\pi i x^2}{p}} e^{-2\pi i k x}\, dx$

$$= \sum_{k\in\mathbb{Z}} e^{-\frac{\pi i k^2 p}{2}} \int_0^p e^{\frac{2\pi i}{p}(x - \frac{kp}{2})^2}\, dx = \gamma_p \int_{-\infty}^{\infty} e^{\frac{2\pi i x^2}{p}}\, dx = \sqrt{p}\,\gamma_p \int_{-\infty}^{\infty} e^{2\pi i u^2}\, du$$

The sum of these tile $\mathbb{R}$ in a couple of ways

$$= e^{\frac{\pi i}{4}} \sqrt{p}\;\gamma_p$$

$$\cancel{= \left(\sum_{k\in\mathbb{Z}} e^{-\frac{\pi i k^2 p}{2}}\right)\left(\int_{-\frac{kp}{2}}^{p-\frac{kp}{2}} e^{\frac{2\pi i u^2}{p}}\, du\right)}$$

Appendix B: Linking pairings

- This is a map $\operatorname{Tor} H_1(M,\mathbb{Z}) \times \operatorname{Tor} H_1(M,\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$

I) U.C.T. $\quad 0 \longrightarrow \operatorname{Ext}(H_1(M,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_2(M),\mathbb{Z}) \longrightarrow 0$

$\Rightarrow \operatorname{Tor} H^2(M,\mathbb{Z}) \cong \operatorname{Ext}(H_1(M,\mathbb{Z}),\mathbb{Z})$.

$\Rightarrow \operatorname{Tor} H_1(M,\mathbb{Z}) \cong \operatorname{Ext}(H_1(M,\mathbb{Z}),\mathbb{Z}) \cong \operatorname{Hom}(\operatorname{Tor} H_1(M,\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$

using Bockstein map

II) Let $\alpha, \beta \in \operatorname{Tor} H_1(M,\mathbb{Z})$.

Then $[n\alpha]=0$ for some n.

Then $n\alpha = \partial S$ for some surface S.

Then $\lambda(\alpha,\beta) = \frac{1}{n}(S\cdot\beta) \mod 1$

- $\mathcal{N}$ = semigroup of all linking pairings

$\mathcal{N} = \bigoplus_p \mathcal{N}_p$ where $\mathcal{N}_p$ is semigroup of linking pairings on p-groups

Thm (Kawauchi, Kojima): $\mathcal{N}_2$ is generated by $A^k(n)$ $k\geq 1$

where $n=1$ $(k=1)$; $n=\pm 1$ $(k\geq 2)$; $n=\pm 1, \pm 5$ $(k\geq 3)$, E_0^k $(k\geq 1)$, E_1^k $(k\geq 2)$

Where $A^k(n) = \frac{n}{2^k} : \mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$

$E_0^k = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix}$ $\quad E_1^k = \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix}$

$k\geq 1$ $\qquad k \geq 2.$

on $(\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k) \times (\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k) \to \mathbb{Q}/\mathbb{Z}$

(8)

They satisfy relations:

$$A^h(n_1) \oplus A^h(n_2) = A^h(n_1+4) \oplus A^h(n_2+4) \qquad h \geq 3$$

$$A^h(n) \oplus 2A^h(-n) = A^h(-n) \oplus E_0^h \qquad h \geq 1$$

$$3A^h(n) = A^h(-n+4) \oplus E_1^h \qquad h \geq 2$$

$$2E_0^h = 2E_1^h \qquad h \geq 2$$

$$A^h(n_1) \oplus A^{h+1}(n_2) = A^h(n_1+2n_2) \oplus A^{h+1}(n_2+2n_1) \qquad h \geq 1$$

$$A^h(n) \oplus E_1^{h+1} = A^h(n+4) \oplus E_0^{h+1} \qquad h \geq 1$$

$$E_1^h \oplus A^{h+1}(n) = E_0^h \oplus A^{h+1}(n+4) \qquad h \geq 2$$

$$A^h(n_1) \oplus A^{h+2}(n_2) = A^h(n_1+4) \oplus A^{h+2}(n_2+4) \qquad h \geq 1.$$