WRT Goal: Define a (2+1)-TQFT from categorical data

1. Use Kirby calc to reduce (2+1) cob category to 1-dim data
2. Define categorical axioms to model Kirby calc
   a) monoidal, braided rigid, pivotal, ribbon, modular / twist non-degen categories
   b) The TQFT this generates.
3. Define universal properties on A such that Mod(A) is modular
   a) ~~Hop~~ bialg, hopf alg, pivotal, ribbon, modular, quasitriangular
4. Construct $U_q(sl_2)$ at root of unity as an example.
   a) h, q, t & versions.
   b) semisimplifications.

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1. Kirby Calculus

a) surgery: $K \hookrightarrow M^3$ closed orientable $N(K)$ tubular nbhd both w/ torus ∂
knot $K := M \setminus N(K)$ knot ext

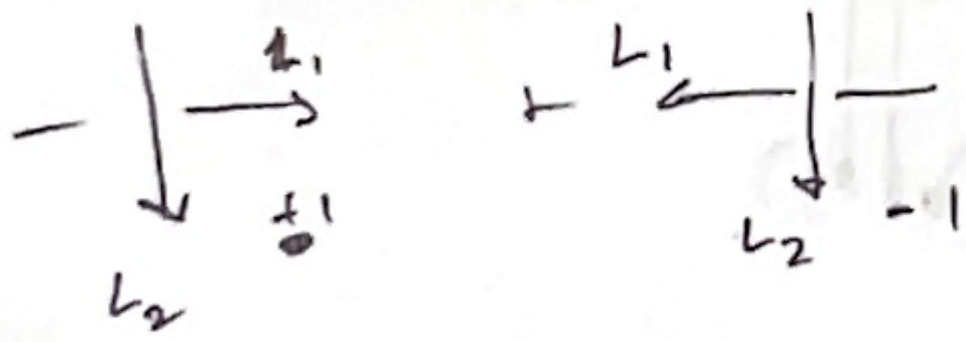
def glue back along $h: \partial D^2 \times S^1 \to \partial K$ homeo is surgery $M_h(K)$


meridian determines the homeo


any curve on ∂K is $p \cdot m + q \cdot l$ $\rightsquigarrow$ $\frac{p}{q} \in \mathbb{Q}$ determines it

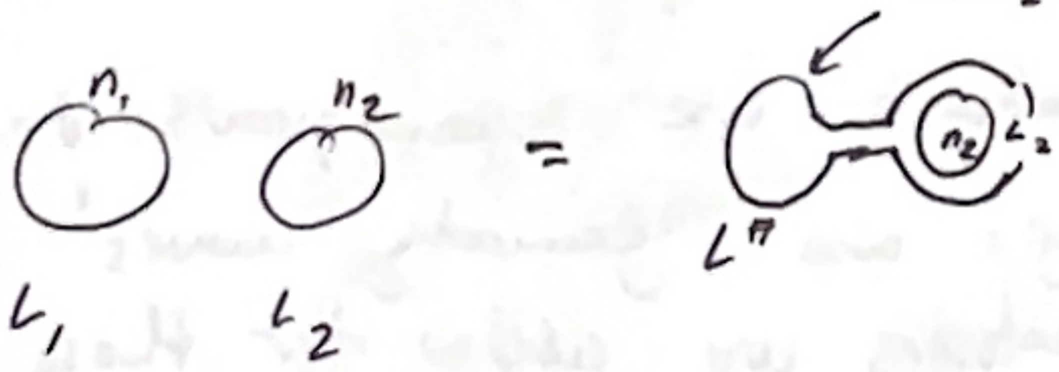
Thm (Lickorish-Wallace) Every closed, orientable 3-man is $S^3_n(K)$ $n \in \mathbb{Z}$
framing on K.

Kirby moves
linking #:



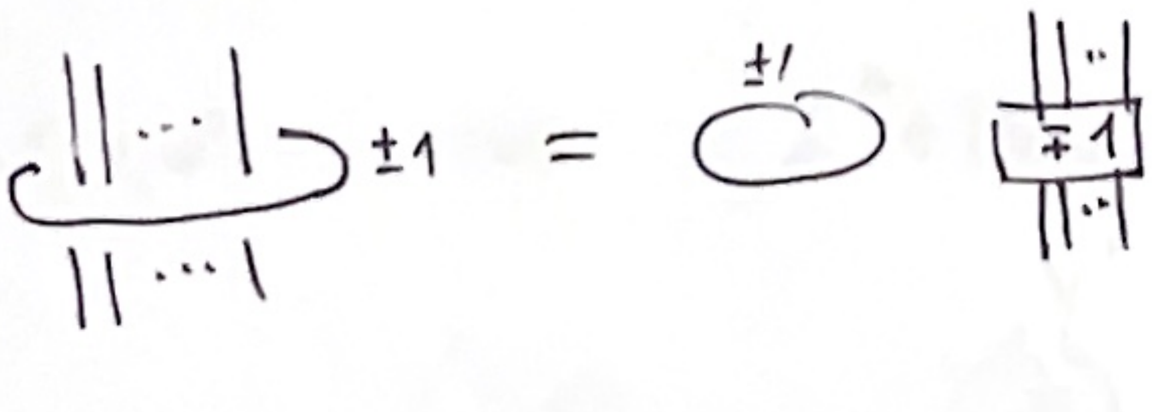
K1: $\mathcal{L} \sqcup \bigcirc^{\pm 1} = \mathcal{L}$

K2:



Thm (Kirby) surgery $S^3_n(\mathcal{L}) \cong_{\text{oriented}} S^3_{n'}(\mathcal{L}')$ iff $\mathcal{L} \overset{\text{Kirby}}{\leftrightarrow} \mathcal{L}'$

Prop (Fenn-Rourke) equivalent set of moves:



2) Modular Categories

Def: Additive category: $\mathrm{Hom}(A,B)$ is a $\mathrm{Vect}_{\mathbb{K}}$, composition is bilinear, there are finite [sums] & in particular a 0 object.

ex: $\mathrm{Vect}_{\mathbb{K}}$, $\mathrm{Mod}(A)$ ($\uparrow$ $\mathbb{K}$Alg), $\mathrm{Rep}_{\mathbb{K}} G$, $\mathrm{Rep}\,\mathfrak{g}$

Def Abelian: ker + coker

Def Semisimple: an object is simple if no nontrivial proper injections; absolutely simple if $\mathrm{End}(V) = \mathbb{K}$ (i.e. it satisfies schur's le[mma])

$\longrightarrow$ all objects are direct sums of simples.

Def Monoidal cat

data:
i) $\otimes : C \times C \longrightarrow C$ functor (bilinear)
ii) $\alpha : (- \otimes -) \otimes - \Leftrightarrow - \otimes (- \otimes -)$ nat trans.
iii) unit 1 object.
$\lambda : 1 \otimes - \Leftrightarrow -$ & $\rho : - \otimes 1 \Leftrightarrow -$

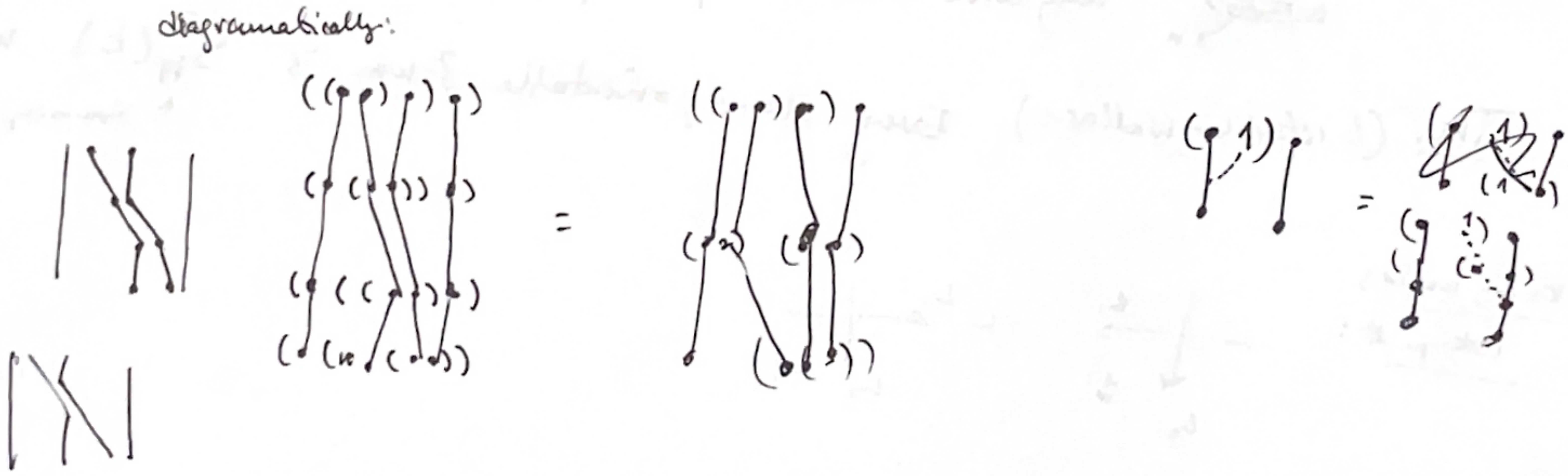
axioms: i) Pentagon:

$$\begin{tikzcd} & ((12)3)4 \arrow[dl] \arrow[ddr] & \\ (1(23))4 \arrow[d] & & \\ 1((23)4) \arrow[dr] & & (12)(34) \arrow[dl] \\ & 1(2(34)) & \end{tikzcd}$$

triangle

$$\begin{tikzcd} (V I)W \arrow[rr] \arrow[dr] & & V(IW) \arrow[dl] \\ & VW & \end{tikzcd}$$

diagramatically:



Thm (MacLane) Parenthesis are unambiguous. & 1 can be freely removed.

idea: Pentagon & triangle are "generating moves" for the restricted class of isotopies we allow for these diagrams.

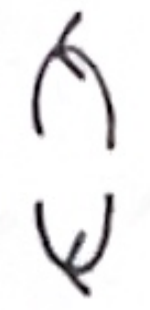
Now pictures like  make sense w/ no confusion.

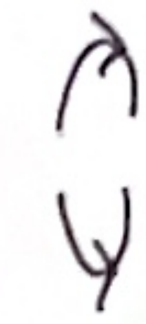
rigid: has left & right duals

data) for each $V \rightsquigarrow V^*$ (left dual) $\qquad {}^*V$ (right dual)

$\overleftarrow{ev}: V^*\otimes V \to 1$ $\qquad \overrightarrow{ev}: V\otimes {}^*V \to 1$

$\overleftarrow{coev}: 1 \to V\otimes V^*$ $\qquad \overrightarrow{coev}: 1 \to {}^*V\otimes V$





axiom) "zig zag"

$V \to 1\otimes V \xrightarrow{\overleftarrow{coev}\otimes \mathrm{id}} V\otimes V^*\otimes V \xrightarrow{\mathrm{id}\otimes\overleftarrow{ev}} V\otimes 1 \to V \quad = \quad V \xrightarrow{\mathrm{id}} V$

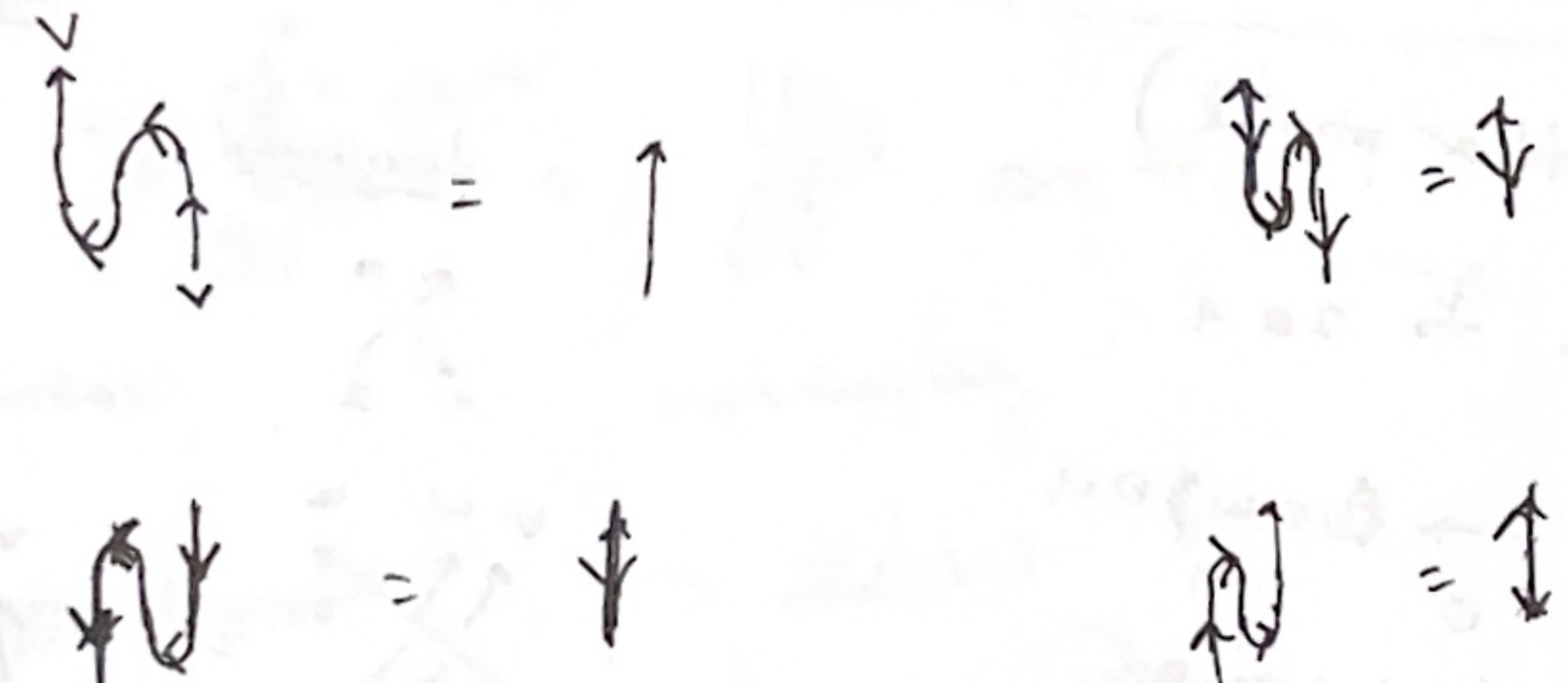
$1\otimes {}^*V \xrightarrow{\overrightarrow{coev}\otimes \mathrm{id}} {}^*V\otimes V\otimes {}^*V \xrightarrow{\mathrm{id}\otimes\overrightarrow{ev}} {}^*V\otimes 1 \quad = \quad {}^*V \xrightarrow{\mathrm{id}} {}^*V$

&

$V^*\otimes 1 \xrightarrow{\mathrm{id}\otimes\overleftarrow{coev}} V^*\otimes V\otimes V^* \xrightarrow{\overleftarrow{ev}\otimes \mathrm{id}} 1\otimes V^* \quad = \quad V^* \xrightarrow{\mathrm{id}} V^*$

$V\otimes 1 \xrightarrow{\mathrm{id}\otimes\overrightarrow{coev}} V\otimes {}^*V\otimes V \xrightarrow{\overrightarrow{ev}\otimes \mathrm{id}} 1\otimes V \quad = \quad V \xrightarrow{\mathrm{id}} V$

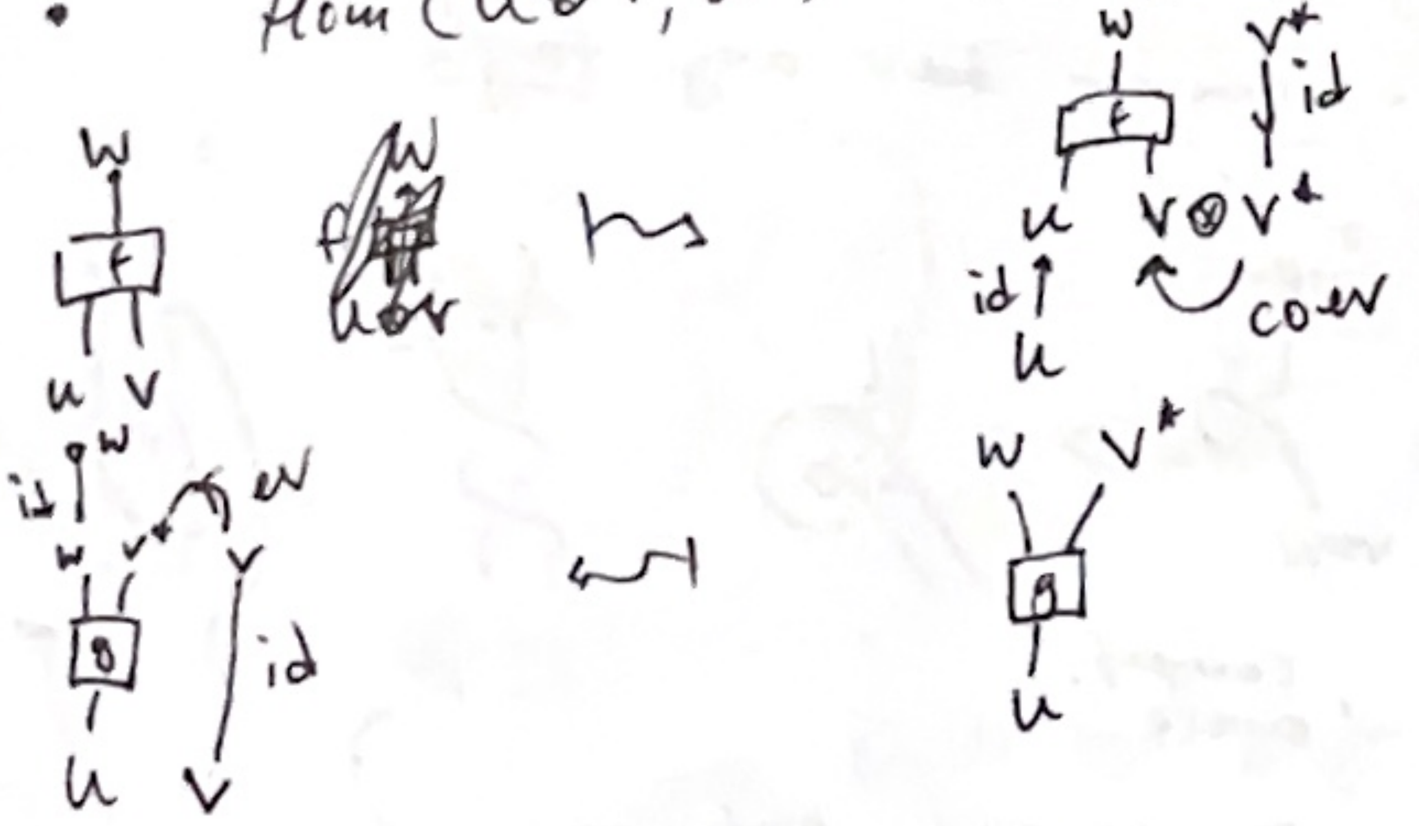
diagramatically



example (vect$_{\mathbb{k}}$) $\quad V^* = \mathrm{Hom}(V,\mathbb{k}) \qquad ev: V^*\otimes V \to \mathbb{k},\ \varphi\otimes v \mapsto \varphi(v) \qquad coev: \mathbb{k} \to V\otimes V^*,\ 1 \mapsto v_i\otimes v_i^*$ (dual basis.)

some consequences:

- $\mathrm{Hom}(U\otimes V, W) \cong \mathrm{Hom}(U, W\otimes V^*)$ & $\mathrm{Hom}(V^*\otimes U, W) = \mathrm{Hom}(U, V\otimes W)$



- $f\in \mathrm{Hom}(V,W) \rightsquigarrow f^*\in\mathrm{Hom}(W^*,V^*)$

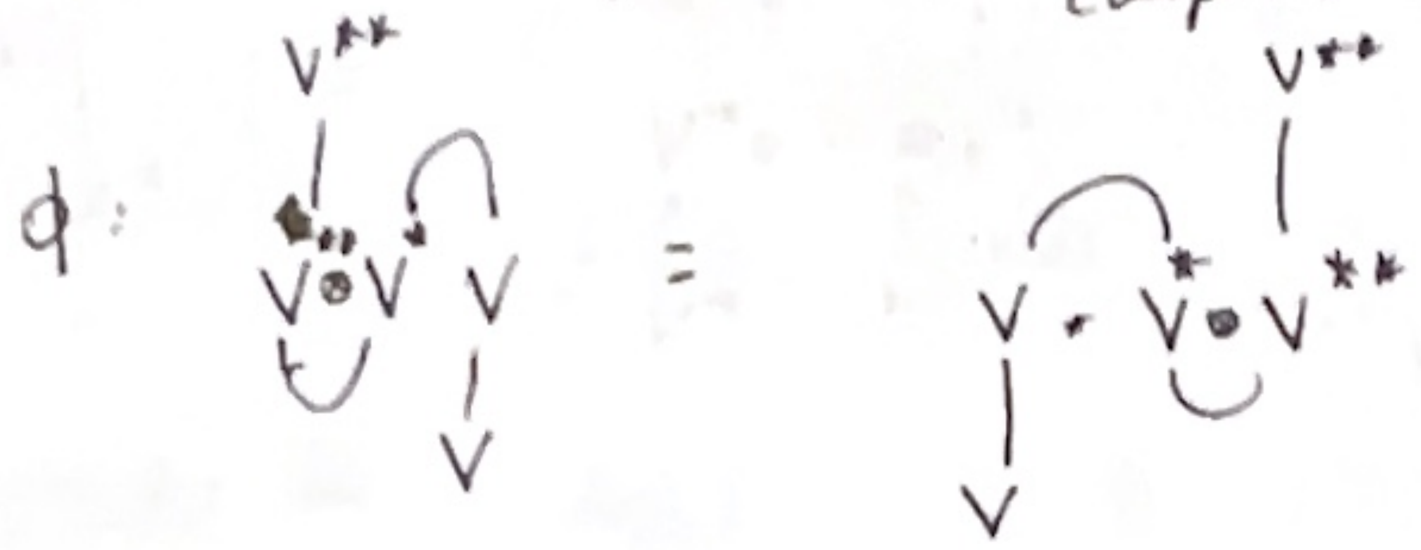
Def compat w/ $\otimes$ if $\overleftarrow{ev}_{V\otimes W} =$

$(V\otimes W)^* \otimes (V\otimes W) \to 1$

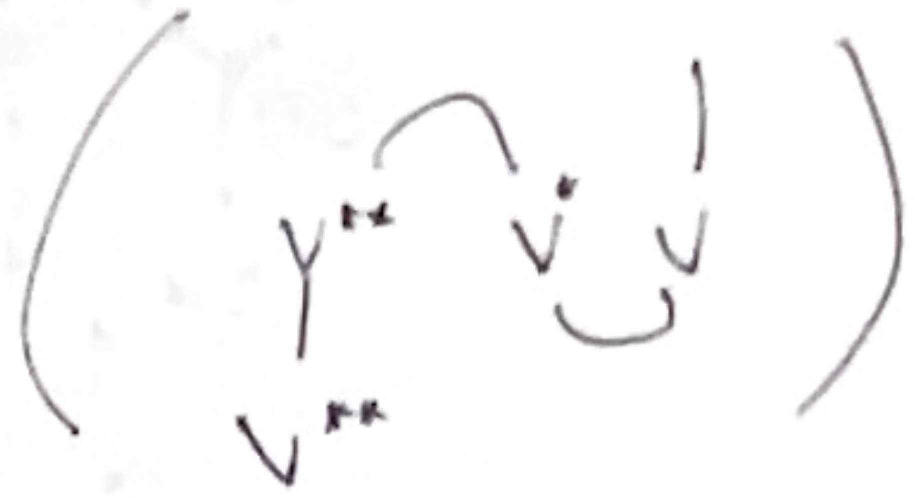
implies $(V\otimes W)^* = W^* \otimes V^*$

& $(f\otimes g)^* = g^* \otimes f^*$

Def Pivotal $V^* = {}^*V$, $f^* = {}^*f$, both compatible



& is invertible



prop Pivotal iff $\phi: - \xrightarrow{\sim} (-)^{**}$ nat iso.

Def Trace$(f) :=$ 

idea: now we have crossingless tangles up to isotopy.

Def Braided (not rigid or pivotal)

data) $c_{1,2}: 1\otimes 2 \xrightarrow{\sim} 2\otimes 1$

diagram

ex: Vect τ

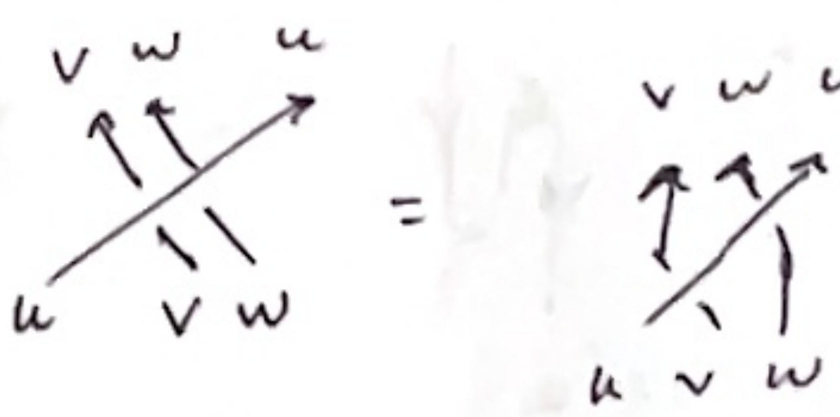
$V\otimes W \to W\otimes V$

$v\otimes w \mapsto w\otimes v$

not interesting b/c symmetric, $\tau^2 = 1$

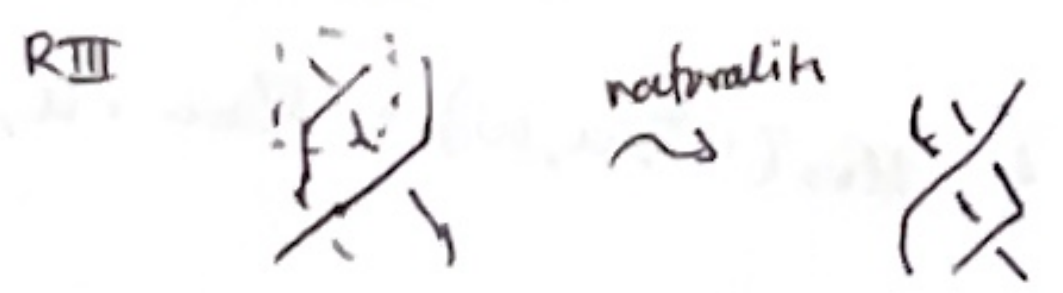
axiom)

$$\begin{tikzcd} u\otimes\{v\otimes w\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \{v\otimes w\}\otimes u \\ v\otimes u\otimes w \arrow[r] & v\otimes w\otimes u \arrow[u] \end{tikzcd}$$



thm (Joyal-Street) morphisms only depend on braid group word.

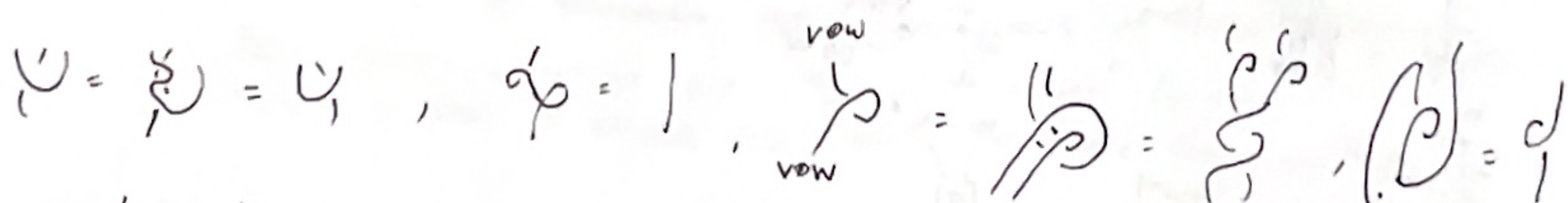
i.e. satisfies braid relations (RII is that c is invertible)



Def Ribbon (Braided + pivotal)

we can now draw any tangle

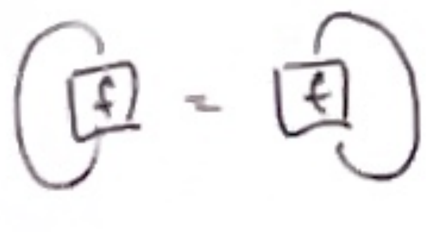
prop



Def $\theta_V =$

or braided w/ compat. duals s.t. w/ $\theta_V: V \to V$ nat isos satisfying & $\theta_{V^*} = \theta_V^*$

(Framed RI)

$=$  (spherical trace)

Modular Category

1) only finitely many simples
2) (s_{ij}) is invertible

$\Longrightarrow$ guarantees

twist nondegen, simples

$\Omega = \sum_i (\dim S_i) S_i$

3) $\Delta_\pm = \Omega \bigcirc \pm 1$ is invertible. $(\neq 0)$

The 3-manifold invariant

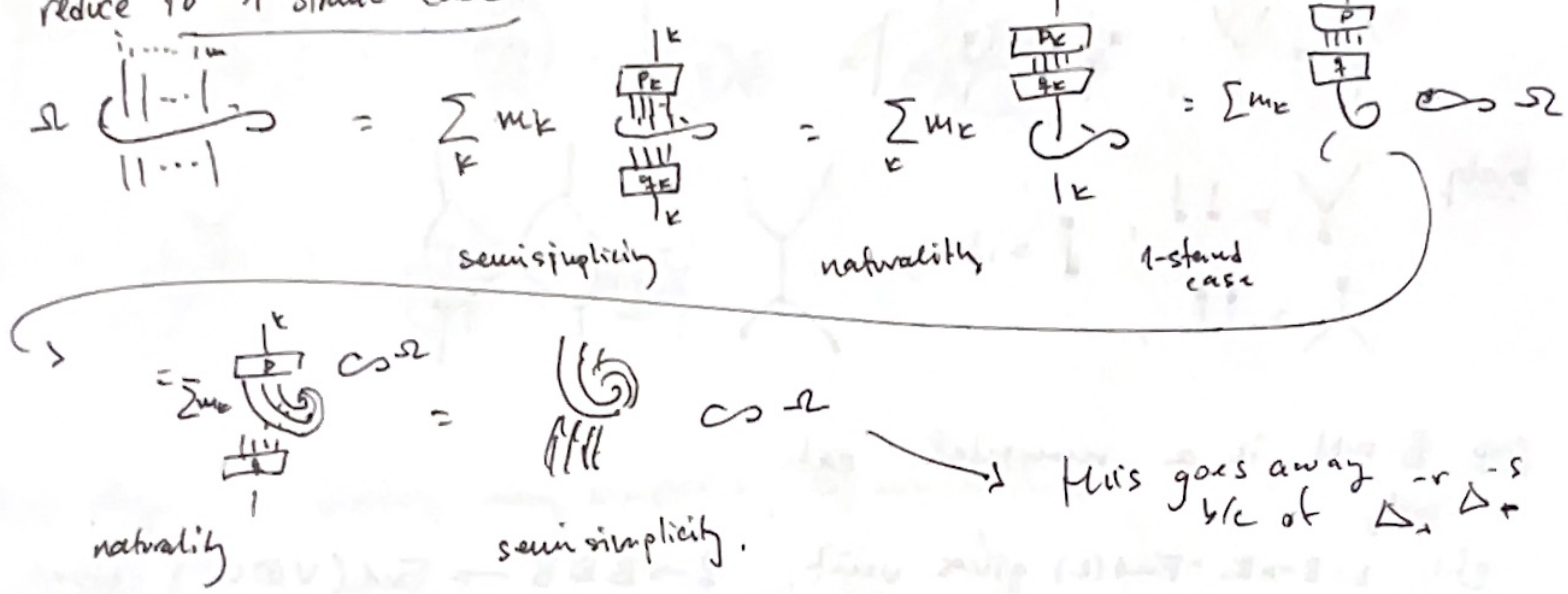
Def if $M = S^3(L)$ framed link $\quad WRT(M) := \dfrac{L_\Omega}{\Delta_+^r \Delta_-^s} \qquad (r,s) = \text{sign}(lk(L))$

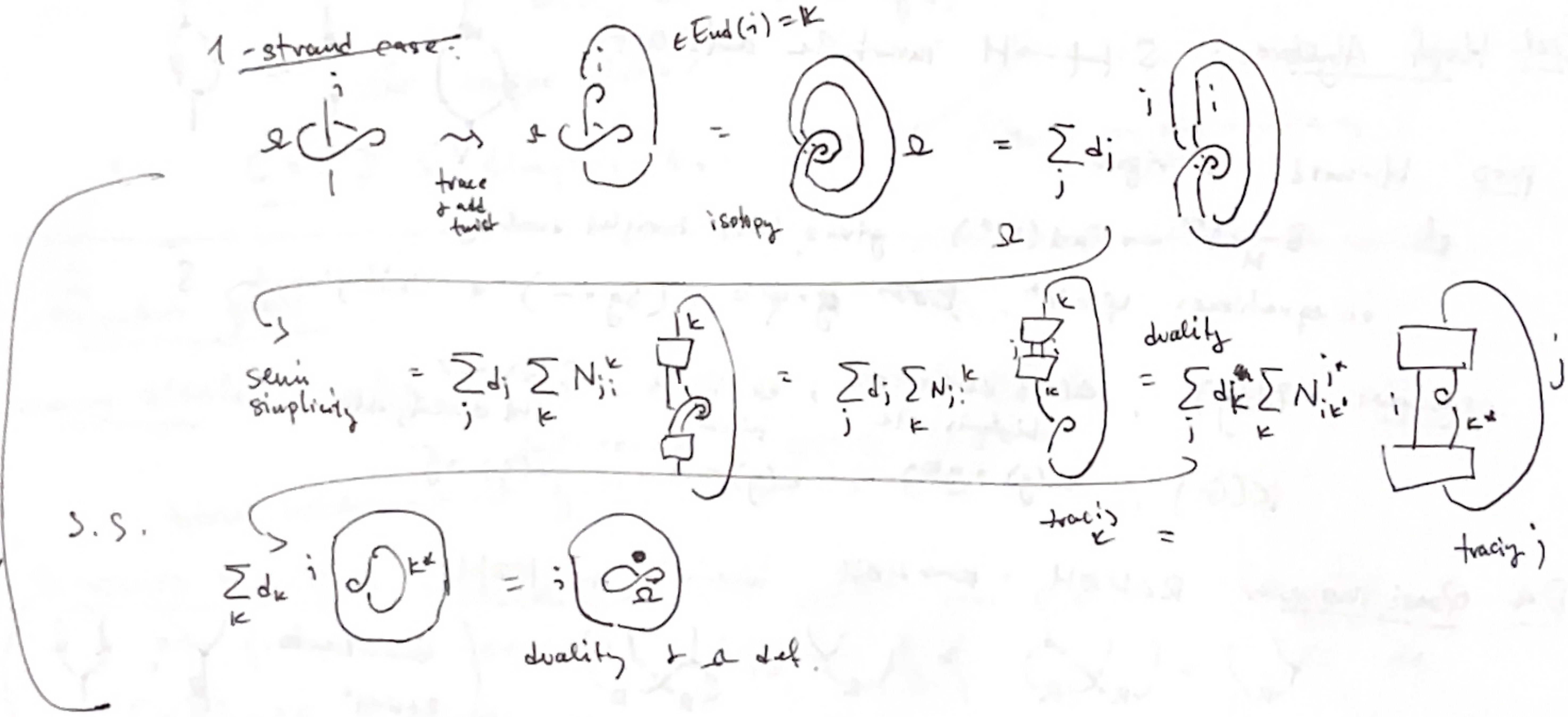
Thm This is a 3-manifold invariant.

pf first orientation invariant b/c duals of simples are simple w/ same dim.

second invariance under FR moves.

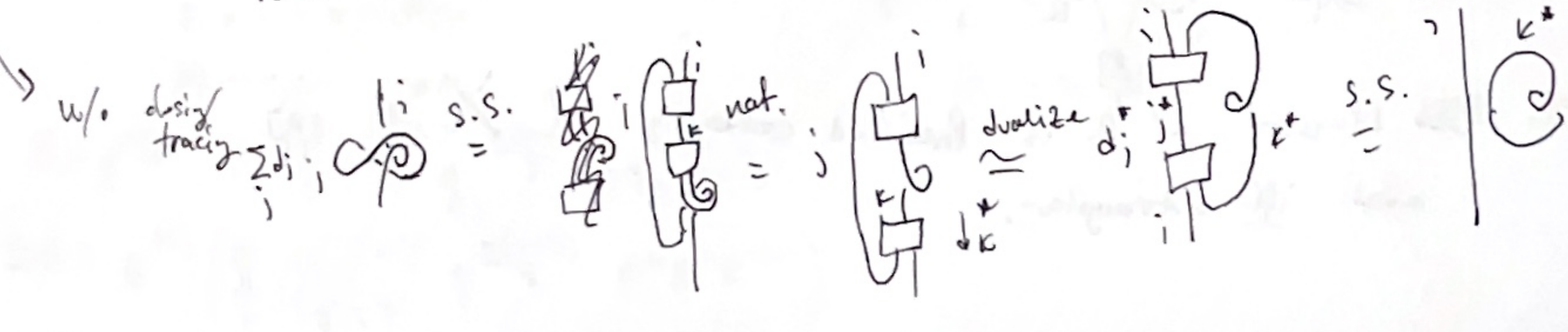
reduce to 1 strand case:





Kirby 1-move:

divide by $\Delta_\pm$ to remove the blow up.



c) The 2-manifold invariant.

$$WRT(\Sigma_g) := \mathrm{Hom}\Big(1, \big(\textstyle\bigoplus_i V_i \otimes V_i^*\big)^{\otimes g}\Big)$$

genus g surface

$WRT(S^3_L \setminus M_g) := WRT\Big($  $\times g \Big)$

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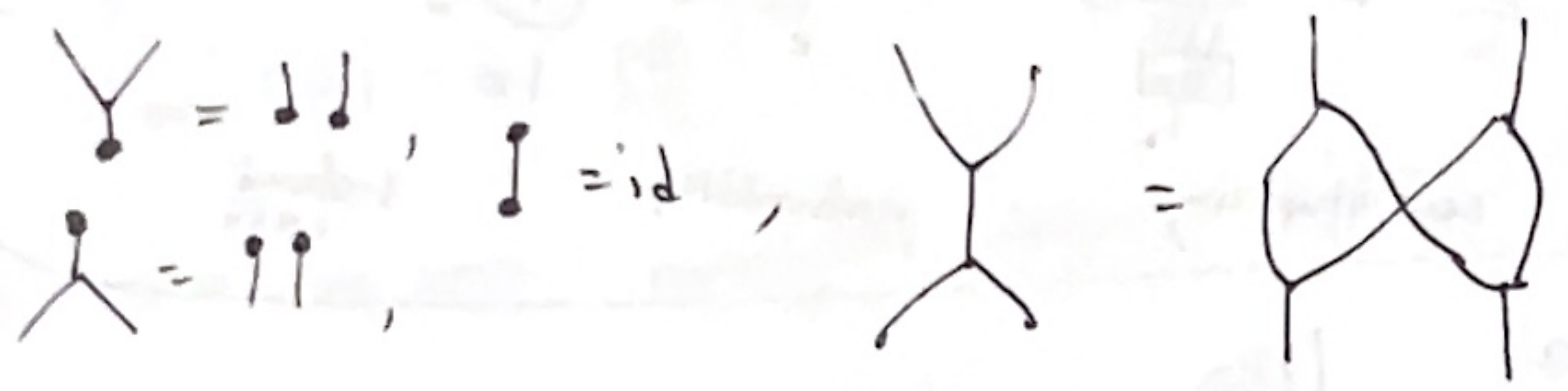
3) Modular Hopf Algebras

Def Algebra. free $\mathbb{K}$-module B w/ $m: B\otimes B \to B$  , $\eta: \mathbb{K} \to B$ 

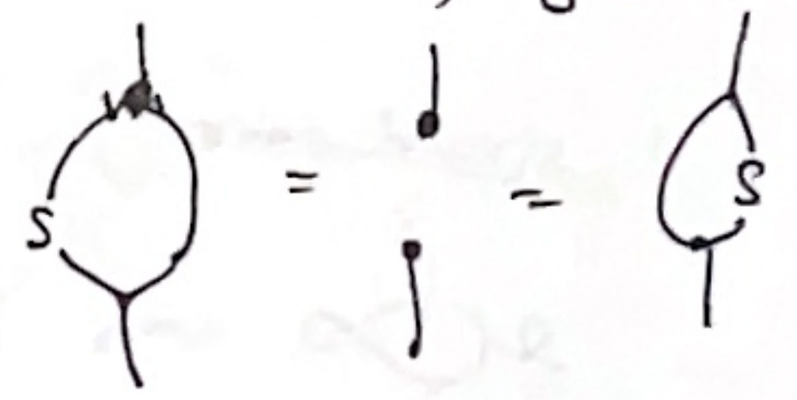
st. 

coalg $\Delta: B \to B\otimes B$  & $\varepsilon: B \to \mathbb{K}$

bialg 

Prop B-mod is a monoidal cat
bialg

pf: $\varepsilon: B \to \mathbb{K} = \mathrm{End}(\mathbb{K})$ gives unit, $B \to B\otimes B \to \mathrm{End}(V\otimes W)$ gives a prod.

Def Hopf Algebra: $S: H \to H$ invertible anti auto 

Prop H-mod is rigid

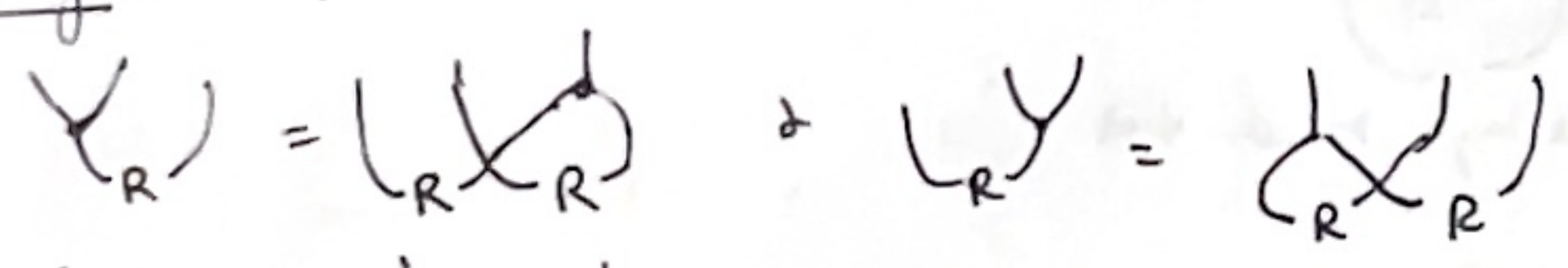
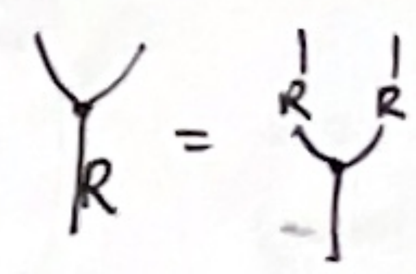
pf $B \xrightarrow[H]{} B^{op} \to \mathrm{End}(V^*)$ gives left & right duality

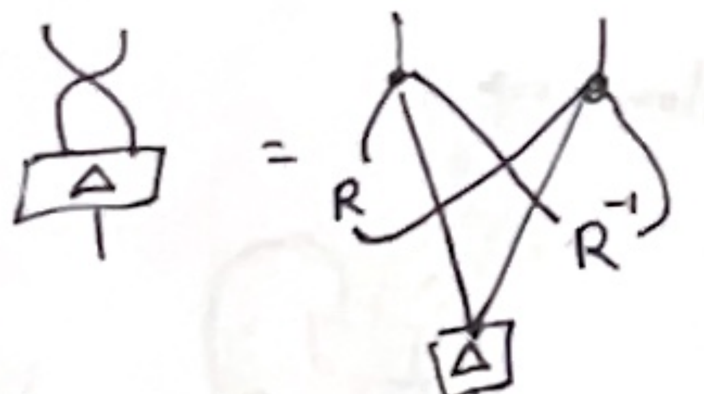
in equations: $\varphi \in V^*$, $g\cdot\varphi = \varphi(Sg\cdot -)$, left given by S^{-1}

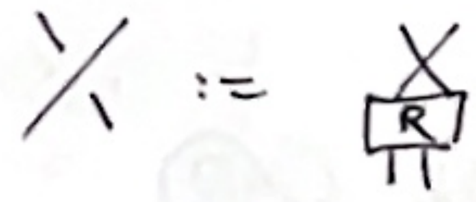
examples $\mathcal{U}(\mathfrak{g})$, $\Delta(x) = x\otimes 1 + 1\otimes x$, $\varepsilon(x) = 0$, $S(x) = -x$
Liebniz rule, trivial, std duality rule.

$\mathbb{K}[G]$, $\Delta(g) = g\otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$

Def Quasi triangular $R \in H\otimes H = \mathbb{K} \to H\otimes H$ invertible in $H\otimes H$

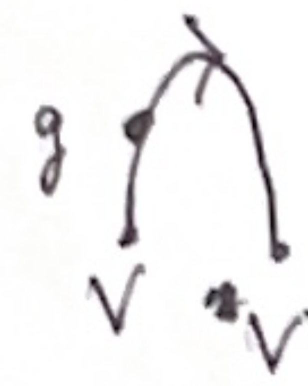
st.  (dualize to $R \in H^*\otimes H^*$ )

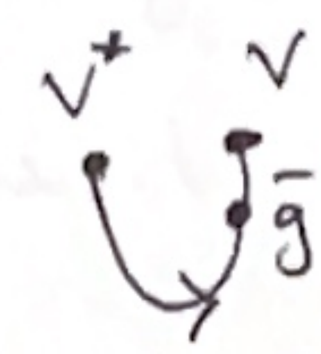
&  (cocommutative up to conj by R)

Prop H-mod w/ R is Braided category w/ 
infact iff q.triangular.

pivotal $\exists \underset{\text{pivot}}{g} \in H$ invertible s.t. $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S^2(x) = g x g^{-1}$
(implies $\otimes$ compatible)

this gives a pivotal category
w/ right $\vec{ev} := \vec{ev}_{\mathbb{K}}(\rho(g) \otimes id_{V^*})$ $\quad \vec{coev} := (id_{V^*} \otimes \rho(g^{-1})) \circ \vec{coev}_{\mathbb{K}}$ ← vectorspace map.





ribbon if

$:= \theta$

gives a ribbon cat.

~~prop gives~~
Modular Hopf alg. finitely many simple +, S_{ij} invertible
why modularity? $\sum_i \theta_i S_{i,j} d_j = v_j^{-1} \dim_q V_j$
has unique solution
& $\underline{C} = \sum v_i^{-1} \dim_q(V_i) d_i \neq 0$ (twist non degeneracy)

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4. Quantum groups

ordinary $\mathcal{U}(sl_2)$ e, f, h $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$
$\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, $S(x) = -x$
trivial braiding ⟶ $g = 1$, $R = 1 \otimes 1$, $\theta = 1$

Powerseries $\mathcal{U}_h(sl_2) = \mathcal{U}_h(sl_2)[[h]]$ over $\mathbb{C}[[h]]$ gen E, H, F
set $q = \exp(h)$, $K = \exp(hH)$ $\quad [E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$
$[H,E] = 2E$, $[H,F] = -2F$

Δ: $E \mapsto 1 \otimes E + E \otimes K$ $\quad \varepsilon: E, F, H \mapsto 0$ $\quad S$: $E \mapsto -EK^{-1}$
$F \mapsto K^{-1} \otimes F + F \otimes 1$ $\quad F \mapsto -KF$
$H \mapsto 1 \otimes H + H \otimes 1$ $\quad H \mapsto -H$

$$R = q^{\frac{H \otimes H}{2}} \left(\sum_n \frac{(q^1 - q^{-1})^{2n}}{(q^n - q^{-n})!} q^{\frac{n(n-1)}{2}} E^n \otimes F^n \right)$$

$$\theta = q^{\frac{H^2}{2}} \left(\sum_n \frac{(q^1 - q^{-1})^{2n}}{(q^n - q^{-n})!} q^{\frac{n(n+3)}{2}} F^n E^n K^{n+1} \right)$$

$\mathcal{U}_q(\mathfrak{sl}_2)$ over $\mathbb{C}(q)$ use $K^{\pm}$ remove H.

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad \Delta(K) = K \otimes K$$

small $\mathcal{U}_\xi(\mathfrak{sl}_2)$ at $\xi^r = 1$

mod $E^r, F^r, K^r - 1$ & set $q = \xi$

Thm $\mathcal{U}_\xi(\mathfrak{sl}_2)$ is a modular Hopf algebra